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Generalizations of Milne's $U(n+1)q$-binomial theorems

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ABSTRACT

In this paper, we use the generalized Andrews–Askey integral and Milne's $U(n+1)q$-binomial theorems to derive some multiple extensions of the q-binomial identities.

1. Introduction

We first recall some definitions, notations and known results in [1,2] which will be used in the following sections. Throughout this paper, it is supposed that $0<|q|<1$. The q-shifted factorials are defined as

$$(a;q)_0=1,\quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\quad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.2}$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In 1846, Heine introduced the $_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$$_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_r \end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_{r+1};q)_n x^n}{(q,b_1,b_2,\ldots,b_r;q)_n}. \tag{1.3}$$

The q-Chu–Vandermonde sums:

$$_2\phi_1\left(\begin{matrix} a,q^{-n} \\ c \end{matrix};q,q\right)=\frac{a^n(c/a;q)_n}{(c;q)_n}. \tag{1.4}$$

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The following is the well-known q-Pfaff–Saalschütz formula:

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q\right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \tag{1.5} $$

Jackson defined the q-integral by [3]

$$ \int_0^d f(t)\mathrm{d}_q t = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n \tag{1.6} $$

and

$$ \int_c^d f(t)\mathrm{d}_q t = \int_0^d f(t)\mathrm{d}_q t - \int_0^c f(t)\mathrm{d}_q t. \tag{1.7} $$

The following is the Andrews–Askey integral [4] which can be derived from Ramanujan's ${}_1\psi_1$:

$$ \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty}\mathrm{d}_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \tag{1.8} $$

provided that no zero factors in the denominator of the integrals. The Andrews–Askey integral is an important formula in hypergeometric series. In [5], the author gives a more general q-integral

$$ \begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q)P_m(\omega, d/b; q)}{(a\omega, b\omega; q)_\infty}\,\mathrm{d}_q\omega &= \frac{t(1-q)(c; q)_n(d; q)_m(q, tq/s, s/t, abst; q)_\infty}{a^n b^m(as, at, bs, bt; q)_\infty} \\ &\quad\times \sum_{k=0}^{n}\frac{(q^{-n}, as, at; q)_k q^k}{(q, c, abst; q)_k}\,{}_3\phi_2\left(\begin{matrix} bs, bt, q^{-m} \\ d, abstq^k \end{matrix}; q, q\right), \end{aligned} \tag{1.9} $$

provided that no zero factors in the denominator of the integrals, where

$$ P_0(a, b; q) = 1, \qquad P_n(a, b; q) = (a-b)(a-bq)\cdots(a-bq^{n-1}), \quad n \geq 1. $$

In [6], Milne proves four terminating $U(n+1)$ refinements of the q-binomial theorem. The following are Eqs. (5.45) and (5.51) of [6]:

Let $x_1, \ldots, x_n$ and z be indeterminate, and let N_i and N be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$. Suppose that none of the denominators in the following identities vanishes. Then

$$ (zq^{-(N_1+\cdots+N_n)}; q)_{N_1+\cdots+N_n} = \sum_{\substack{0\leq y_i\leq N_i,\\ i=1,2,\ldots,n}}\left\{\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{(q\frac{x_r}{x_s}; q)_{y_r}}\right]q^{y_2+2y_3+\cdots+(n-1)y_n}z^{y_1+y_2+\cdots+y_n}\right\} \tag{1.10} $$

and

$$ \begin{aligned} (zq^{-N}; q)_N &= \sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}}\left\{\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\left[(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N}; q)_{y_1+\cdots+y_n}\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}\right]\right. \\ &\quad\left.\times\left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2\cdots,y_n)}\right]\prod_{i=1}^{n}[(x_i)^{ny_i-(y_1+\cdots+y_n)}]z^{y_1+y_2+\cdots+y_n}\right\}, \end{aligned} \tag{1.11} $$

where $e_2(y_1, y_2, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Milne also give the following second version of (1.11) [6]:

$$ \begin{aligned} (zq^{-N}; q)_N &= \sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}}\left\{\prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right](q^{-N}; q)_{y_1+\cdots+y_n}\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}\right. \\ &\quad\left.\times\, q^{y_2+2y_3+\cdots+(n-1)y_n}z^{y_1+y_2+\cdots+y_n}\right\}. \end{aligned} \tag{1.12} $$

Mainly inspired by the papers [7–13], we use the extension of the Andrews–Askey integral (1.9) to derive some multiple identities, which could be thought of the extensions of Milne's $U(n+1)q$-binomial theorem (1.10)–(1.12).

2. The extensions of the formula (1.10)

The following theorem could be thought of the extensions of Milne's $U(n+1)$ refinement of q-binomial theorem (1.10).

Theorem 2.1. *Let $x_1, \ldots, x_n$ and a, c, s, t be indeterminate, and let m, N_i be nonnegative integers for $i = 1, 2, \ldots, n$ with $n \geq 1$, then we have*

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ c, ast \end{matrix}; q, q\right) &= \frac{(astq^{-(N_1+\cdots+N_n)}; q)_{N_1+\cdots+N_n}}{(sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}; q)_{N_1+\cdots+N_n}} \\
&\times \sum_{\substack{0\leq y_i\leq N_i,\\ i=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{\left(q\frac{x_r}{x_s}; q\right)_{y_r}}\right] [q^{y_2+2y_3+\cdots+(n-1)y_n+(y_1+y_2+\cdots+y_n)(N_1+\cdots+N_n)}] \right. \\
&\left. \times \sum_{k=0}^{m} \frac{(q^{-m}, as, at; q)_k q^k}{(q, c, astq^{-(N_1+\cdots+N_n)}; q)_k} {}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-(N_1+\cdots+N_n)} \end{matrix}; q, q\right)\right\},
\end{aligned}
\tag{2.1}
$$

provided that no zero factors in the denominator.

Proof. Rewriting (1.10) as

$$
\frac{1}{(z;q)_\infty} = \sum_{\substack{0\leq y_i\leq N_i,\\ i=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{\left(q\frac{x_r}{x_s}; q\right)_{y_r}}\right] q^{y_2+2y_3+\cdots+(n-1)y_n} \frac{z^{y_1+y_2+\cdots+y_n}}{(zq^{-(N_1+\cdots+N_n)}; q)_\infty}\right\}. \tag{2.2}
$$

Multiplying Eq. (2.2) by

$$
\frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az; q)_\infty},
$$

then we obtain

$$
\begin{aligned}
\frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az, z; q)_\infty} &= \sum_{\substack{0\leq y_i\leq N_i,\\ i=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{\left(q\frac{x_r}{x_s}; q\right)_{y_r}}\right] q^{y_2+2y_3+\cdots+(n-1)y_n} \right. \\
&\left. \times \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q) z^{y_1+y_2+\cdots+y_n}}{(az, zq^{-(N_1+\cdots+N_n)}; q)_\infty} \right\}.
\end{aligned}
\tag{2.3}
$$

Taking the q-integral on both sides of the above identity with respect to variable z gives

$$
\begin{aligned}
\int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az, z; q)_\infty} \mathrm{d}_q z &= \sum_{\substack{0\leq y_i\leq N_i,\\ i=1,2,\ldots,n}} \left\{ \prod_{1\leq r<s\leq n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{\left(q\frac{x_r}{x_s}; q\right)_{y_r}}\right] q^{y_2+2y_3+\cdots+(n-1)y_n} \right. \\
&\left. \times \int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q) z^{y_1+y_2+\cdots+y_n}}{(az, zq^{-(N_1+\cdots+N_n)}; q)_\infty} \mathrm{d}_q z \right\}.
\end{aligned}
\tag{2.4}
$$

Using (1.9), we get

$$
\int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az, z; q)_\infty} \mathrm{d}_q z = \frac{t(1-q)(c;q)_m (q, tq/s, s/t, ast; q)_\infty}{a^m (as, at, s, t; q)_\infty} {}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ c, ast \end{matrix}; q, q\right) \tag{2.5}
$$

and

$$
\begin{aligned}
&\int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q) z^{y_1+y_2+\cdots+y_n}}{(az, zq^{-(N_1+\cdots+N_n)}; q)_\infty} \mathrm{d}_q z \\
&= \frac{t(1-q)(c;q)_m (q, tq/s, s/t, astq^{-(N_1+\cdots+N_n)}; q)_\infty}{a^m q^{-(y_1+y_2+\cdots+y_n)(N_1+\cdots+N_n)} (as, at, sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}; q)_\infty} \sum_{k=0}^{m} \left\{ \frac{(q^{-m}, as, at; q)_k q^k}{(q, c, astq^{-(N_1+\cdots+N_n)}; q)_k} \right. \\
&\left. \times {}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-(N_1+\cdots+N_n)} \end{matrix}; q, q\right)\right\}.
\end{aligned}
\tag{2.6}
$$

Substituting (2.5) and (2.6) into (2.4), we obtain (2.1). □

Remark 2.2. If $at = 1$ in (2.1), we get Milne's $U(n+1)$ q-binomial theorem (1.10).

As applications of Theorem 2.1, we get the following results.

Corollary 2.3. *Let $x_1, \dots, x_n$ and a, b, c be indeterminate, and let N_i be nonnegative integers for $i = 1, 2, \dots, n$ with $n \geq 1$, then we have*

$$\sum_{\substack{0\le y_i\le N_i,\\ i=1,2,\dots,n}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{\left(q\frac{x_r}{x_s};q\right)_{y_r}}\right][q^{y_2+2y_3+\dots+(n-1)y_n+(y_1+y_2+\dots+y_n)(N_1+\dots+N_n)}] \right.$$
$$\left. \times\, {}_3\phi_2\left(\begin{matrix} a, b, q^{-(y_1+y_2+\dots+y_n)} \\ 0, c \end{matrix}; q, q\right)\right\}$$
$$= \frac{(a,b;q)_{N_1+\dots+N_n}}{(c;q)_{N_1+\dots+N_n}}, \tag{2.7}$$

provided that no zero factors in the denominator.

Proof. Let $m = 0$ in (2.1), we get

$$\sum_{\substack{0\le y_i\le N_i,\\ i=1,2,\dots,n}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{\left(q\frac{x_r}{x_s};q\right)_{y_r}}\right][q^{y_2+2y_3+\dots+(n-1)y_n+(y_1+y_2+\dots+y_n)(N_1+\dots+N_n)}] \right.$$
$$\left. \times\, {}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)}, q^{-(y_1+y_2+\dots+y_n)} \\ 0, astq^{-(N_1+\dots+N_n)} \end{matrix}; q, q\right)\right\}$$
$$= \frac{(sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}{(astq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}, \tag{2.8}$$

Replacing $(sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)}, astq^{-(N_1+\dots+N_n)})$ by (a, b, c), we obtain (2.7). □

If $n = 1$ in (2.7), we get the following somewhat new identity, which includes the terminating case of the q-binomial theorem as special case.

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^{kn}}{(q;q)_k}\,{}_3\phi_2\left(\begin{matrix} a, b, q^{-k} \\ 0, c \end{matrix}; q, q\right) = \frac{(a,b;q)_n}{(c;q)_n}. \tag{2.9}$$

If $a = c$ in (2.9), we get the terminating case of the q-binomial theorem.

Corollary 2.4. *Let $x_1, \dots, x_n$ and a, c, s, t be indeterminate, and let m, N_i be nonnegative integers for $i = 1, 2, \dots, n$ with $n \geq 1$, then we have*

$$\sum_{\substack{0\le y_i\le N_i,\\ i=1,2,\dots,n}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{\left(q\frac{x_r}{x_s};q\right)_{y_r}}\right][q^{y_2+2y_3+\dots+(n-1)y_n+(y_1+y_2+\dots+y_n)(N_1+\dots+N_n)}] \right.$$
$$\left. \times \sum_{k=0}^{m}\left[\frac{(q^{-m}, at;q)_k q^k}{(q, ctq^{-(N_1+\dots+N_n)};q)_k}\,{}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)}, q^{-(y_1+y_2+\dots+y_n)} \\ 0, ctq^{k-(N_1+\dots+N_n)} \end{matrix}; q, q\right)\right]\right\}$$
$$= \frac{(at)^m(c/a;q)_m(sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}{(ct;q)_m(ctq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}, \tag{2.10}$$

provided that no zero factors in the denominator.

Proof. Let $c = as$ in (2.1) and use the q-Chu–Vandermonde sums (1.4), we obtain (2.10). □

Corollary 2.5. *Let $x_1, \dots, x_n$ and a, s, t be indeterminate, and let m, N_i be nonnegative integers for $i = 1, 2, \dots, n$ with $n \geq 1$, then we have*

$$\sum_{\substack{0\le y_i\le N_i,\\ i=1,2,\dots,n}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s};q)_{y_r}}{\left(q\frac{x_r}{x_s};q\right)_{y_r}}\right][q^{y_2+2y_3+\dots+(n-1)y_n+(y_1+y_2+\dots+y_n)(N_1+\dots+N_n)}] \right.$$
$$\left. \times \sum_{k=0}^{m}\frac{(q^{-m}, as, at;q)_k q^k}{(q, aq^{1-m}, astq^{-(N_1+\dots+N_n)};q)_k}\,{}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)}, q^{-(y_1+y_2+\dots+y_n)} \\ 0, astq^{k-(N_1+\dots+N_n)} \end{matrix}; q, q\right)\right\}$$
$$= \frac{(s,t;q)_m(sq^{-(N_1+\dots+N_n)}, tq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}{(ast, 1/a;q)_m(astq^{-(N_1+\dots+N_n)};q)_{N_1+\dots+N_n}}, \tag{2.11}$$

provided that no zero factors in the denominator.

Proof. Let $c = aq^{1-m}$ in (2.1), we get

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ aq^{1-m}, ast \end{matrix}; q, q\right) &= \frac{(astq^{-(N_1+\cdots+N_n)}; q)_{N_1+\cdots+N_n}}{(sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}; q)_{N_1+\cdots+N_n}} \\
&\times \sum_{\substack{0\le y_i\le N_i,\\ i=1,2,\ldots,n}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n}\left[\frac{(\frac{x_r}{x_s}q^{-N_s}; q)_{y_r}}{\left(q\frac{x_r}{x_s}; q\right)_{y_r}}\right] [q^{y_2+2y_3+\cdots+(n-1)y_n+(y_1+y_2+\cdots+y_n)(N_1+\cdots+N_n)}] \right. \\
&\times \left. \sum_{k=0}^{m} \frac{(q^{-m}, as, at; q)_k q^k}{(q, aq^{1-m}, astq^{-(N_1+\cdots+N_n)}; q)_k} {}_3\phi_2\left(\begin{matrix} sq^{-(N_1+\cdots+N_n)}, tq^{-(N_1+\cdots+N_n)}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-(N_1+\cdots+N_n)} \end{matrix}; q, q\right)\right\}.
\end{aligned}
\tag{2.12}
$$

Using the well-known q-Pfaff–Saalschütz formula (1.5), we have

$$
{}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ aq^{1-m}, ast \end{matrix}; q, q\right) = \frac{(s, t; q)_m}{(ast, 1/a; q)_m}.
\tag{2.13}
$$

Substituting (2.13) into (2.12), we obtain (2.11). $\square$

3. The extensions of the formula (1.11)

The following theorem could be thought of the extensions of Milne's $U(n+1)$ refinement of q-binomial theorem (1.11).

Theorem 3.1. *Let $x_1, \ldots, x_n$ with $n \ge 1$ and a, c, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$
\begin{aligned}
{}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ c, ast \end{matrix}; q, q\right) &= \frac{(astq^{-N}; q)_N}{(sq^{-N}, tq^{-N}; q)_N} \sum_{\substack{y_1,\ldots,y_n\ge 0,\\ 0\le y_1+\cdots+y_n\le N}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \right. \\
&\times [(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N}; q)_{y_1+\cdots+y_n}][q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots\binom{y_n}{2}]-e_2(y_1,y_2\cdots,y_n)+N(y_1+\cdots+y_n)}] \\
&\times \left. \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1} \cdot \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)} \sum_{k=0}^{m}\left\{ \frac{(q^{-m}, as, at; q)_k q^k}{(q, c, astq^{-N}; q)_k} {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-N} \end{matrix}; q, q\right)\right\}\right\},
\end{aligned}
\tag{3.1}
$$

provided that no zero factors in the denominator, where $e_2(y_1, y_2, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Proof. Rewriting (1.11) as

$$
\begin{aligned}
\frac{1}{(z; q)_\infty} &= \sum_{\substack{y_1,\ldots,y_n\ge 0,\\ 0\le y_1+\cdots+y_n\le N}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \left[(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}\right] \right. \\
&\times \left. [q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)[\binom{y_1}{2}+\cdots\binom{y_n}{2}]-e_2(y_1,y_2\cdots,y_n)}] \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)} \frac{z^{y_1+y_2+\cdots+y_n}}{(zq^{-N}; q)_\infty}\right\}.
\end{aligned}
\tag{3.2}
$$

Multiplying Eq. (3.2) by

$$
\frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az; q)_\infty}
$$

and taking the q-integral on both sides of the above identity with respect to variable z, then we obtain

$$
\begin{aligned}
\int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q)}{(az, z; q)_\infty}\, \mathrm{d}_q z &= \sum_{\substack{y_1,\ldots,y_n\ge 0,\\ 0\le y_1+\cdots+y_n\le N}} \left\{ \prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \right. \\
&\times \left[(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}\right] \left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2\cdots,y_n)}\right] \\
&\times \left. \left[\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}\right] \int_s^t \frac{(qz/s, qz/t; q)_\infty P_m(z, c/a; q) z^{y_1+y_2+\cdots+y_n}}{(az, zq^{-N}; q)_\infty}\, \mathrm{d}_q z\right\}.
\end{aligned}
\tag{3.3}
$$

Using (1.9), (3.3) can be rewritten as (3.1). $\square$

Remark 3.2. If $at = 1$ in (3.1), we get Milne's $U(n+1)$ q-binomial theorem (1.11).

As applications of Theorem 3.1, we get the following results.

Corollary 3.3. *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, b be indeterminate, then for any nonnegative integer N, we have*

$$\sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] [(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N};q)_{y_1+\cdots+y_n}] \right.$$
$$\times \left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2,\ldots,y_n)+N(y_1+\cdots+y_n)}\right] \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \cdot \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}$$
$$\left. \times\, {}_3\phi_2\left(\begin{matrix} a, b, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, c \end{matrix}; q, q\right)\right\} = \frac{(a,b;q)_N}{(c;q)_N}, \tag{3.4}$$

provided that no zero factors in the denominator.

Proof. Let $m = 0$ in (3.1), we get

$$\frac{(sq^{-N}, tq^{-N};q)_N}{(astq^{-N};q)_N} = \sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] [(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N};q)_{y_1+\cdots+y_n}] \right.$$
$$\times \left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2,\ldots,y_n)+N(y_1+\cdots+y_n)}\right] \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \cdot \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}$$
$$\left. \times\, {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{-N} \end{matrix}; q, q\right)\right\}. \tag{3.5}$$

Replacing $(sq^{-N}, tq^{-N}, astq^{-N})$ by (a, b, c) in (3.5), we obtain (3.4). $\square$

If $n = 1$ in (3.4), we get (2.9) again.

Corollary 3.4. *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, c, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$\sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] [(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N};q)_{y_1+\cdots+y_n}] \right.$$
$$\times \left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2,\ldots,y_n)+N(y_1+\cdots+y_n)}\right] \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \cdot \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}$$
$$\left. \times \sum_{k=0}^{m}\left\{\frac{(q^{-m}, at;q)_k q^k}{(q, ctq^{-N};q)_k}\, {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, ctq^{k-N} \end{matrix}; q, q\right)\right\}\right\}$$
$$= \frac{(at)^m (c/a;q)_m (sq^{-N}, tq^{-N};q)_N}{(ct;q)_m (ctq^{-N};q)_N}, \tag{3.6}$$

provided that no zero factors in the denominator, where $e_2(y_1, y_2, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Proof. Let $c = as$ in (3.1) and use the q-Chu–Vandermonde sums (1.4), we obtain (3.6). $\square$

Corollary 3.5. *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$\sum_{\substack{y_1,\ldots,y_n\geq 0,\\ 0\leq y_1+\cdots+y_n\leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] [(-1)^{(n-1)(y_1+\cdots+y_n)}(q^{-N};q)_{y_1+\cdots+y_n}] \right.$$
$$\times \left[q^{y_2+2y_3+\cdots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\cdots\binom{y_n}{2}\right]-e_2(y_1,y_2,\ldots,y_n)+N(y_1+\cdots+y_n)}\right] \prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \cdot \prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\cdots+y_n)}$$
$$\left. \times \sum_{k=0}^{m}\left\{\frac{(q^{-m}, as, at;q)_k q^k}{(q, c, astq^{-N};q)_k}\, {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-N} \end{matrix}; q, q\right)\right\}\right\}$$

$$= \frac{(s, t; q)_m (sq^{-N}, tq^{-N}; q)_N}{(ast, 1/a; q)_m (astq^{-N}; q)_N}, \tag{3.7}$$

provided that no zero factors in the denominator.

Proof. Letting $c = aq^{1-m}$ in (3.1) and using the well-known q-Pfaff–Saalschütz formula (1.5) we get (3.7). □

Similarly, if we apply the Andrews–Askey integral to (1.12), we get the following second versions of (3.1), (3.4), (3.6) and (3.7).

Theorem 3.6 (*Second version of* (3.1)). *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, c, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$\begin{aligned}
{}_3\phi_2\left(\begin{matrix} as, at, q^{-m} \\ c, ast \end{matrix}; q, q\right) &= \frac{(astq^{-N}; q)_N}{(sq^{-N}, tq^{-N}; q)_N} \sum_{\substack{y_1,\ldots,y_n \geq 0, \\ 0 \leq y_1+\cdots+y_n \leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right] \right. \\
&\times \left[(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q \right)_{y_r}^{-1} \right] [q^{y_2+2y_3+\cdots+(n-1)y_n+N(y_1+\cdots+y_n)}] \\
&\times \left. \sum_{k=0}^{m} \left\{ \frac{(q^{-m}, as, at; q)_k q^k}{(q, c, astq^{-N}; q)_k} {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-N} \end{matrix}; q, q\right) \right\} \right\},
\end{aligned} \tag{3.8}$$

provided that no zero factors in the denominator.

Remark 3.7. If $at = 1$ in (3.8), we get Milne's $U(n+1)$ q-binomial theorem (1.12).

Corollary 3.8 (*Second version of* (3.4)). *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, b, c be indeterminate, then for any nonnegative integers N, we have*

$$\begin{aligned}
&\sum_{\substack{y_1,\ldots,y_n \geq 0, \\ 0 \leq y_1+\cdots+y_n \leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right] \left[(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q \right)_{y_r}^{-1} \right] \right. \\
&\quad \times \left. [q^{y_2+2y_3+\cdots+(n-1)y_n+N(y_1+\cdots+y_n)}] {}_3\phi_2\left(\begin{matrix} a, b, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, c \end{matrix}; q, q\right) \right\} \\
&= \frac{(a, b; q)_N}{(c; q)_N},
\end{aligned} \tag{3.9}$$

provided that no zero factors in the denominator.

Corollary 3.9 (*Second version of* (3.6)). *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$\begin{aligned}
&\sum_{\substack{y_1,\ldots,y_n \geq 0, \\ 0 \leq y_1+\cdots+y_n \leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right] \left[(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q \right)_{y_r}^{-1} \right] \right. \\
&\quad \times \left. [q^{y_2+2y_3+\cdots+(n-1)y_n+N(y_1+\cdots+y_n)}] \sum_{k=0}^{m} \left\{ \frac{(q^{-m}, at; q)_k q^k}{(q, ctq^{-N}; q)_k} {}_3\phi_2\left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, ctq^{k-N} \end{matrix}; q, q\right) \right\} \right\} \\
&= \frac{(at)^m (c/a; q)_m (sq^{-N}, tq^{-N}; q)_N}{(ct; q)_m (ctq^{-N}; q)_N},
\end{aligned} \tag{3.10}$$

provided that no zero factors in the denominator.

Corollary 3.10 (*Second version of* (3.7)). *Let $x_1, \ldots, x_n$ with $n \geq 1$ and a, s, t be indeterminate, then for any nonnegative integers m and N, we have*

$$\sum_{\substack{y_1,\ldots,y_n \geq 0, \\ 0 \leq y_1+\cdots+y_n \leq N}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1 - \frac{x_r}{x_s} q^{y_r - y_s}}{1 - \frac{x_r}{x_s}} \right] \left[(q^{-N}; q)_{y_1+\cdots+y_n} \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q \right)_{y_r}^{-1} \right] \right.$$

$$\times \ [q^{y_2+2y_3+\cdots+(n-1)y_n+N(y_1+\cdots+y_n)}] \sum_{k=0}^{m} \left\{ \frac{(q^{-m}, as, at; q)_k q^k}{(q, aq^{1-m}, astq^{-N}; q)_k} {}_3\phi_2 \left(\begin{matrix} sq^{-N}, tq^{-N}, q^{-(y_1+y_2+\cdots+y_n)} \\ 0, astq^{k-N} \end{matrix} ; q, q \right) \right\} \Bigg\}$$

$$= \frac{(s, t; q)_m (sq^{-N}, tq^{-N}; q)_N}{(ast, 1/a; q)_m (astq^{-N}; q)_N}, \tag{3.11}$$

provided that no zero factors in the denominator.

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